Extensions of Alspach's theorem to regular multipartite tournaments

Yubao Guo

Institute for Mathematics of Information Processing

RWTH Aachen University, 52056 Aachen, Germany

June 20, 2022

1. **Terminology and notation**
2. **Semicomplete digraphs, tournaments**
   - Rédei's theorem and Camion's theorem
   - Moon's theorem and Alspach's theorem
3. **Semicomplete multipartite digraphs, c-partite tournaments**
   - k-cycle in semicomplete multipartite digraphs
   - $(c+1)$-cycles in c-partite tournaments
4. **Extensions of Alspach's theorem to multipartite tournaments**
   - Extension I: quasi_{p}-arc-pancyclic regular tournaments
   - Extension II: quasi_{l}-arc-pancyclic regular tournaments
   - Extension III: quasi_{o}-arc-pancyclic regular tournaments
   - Extension IV: quasi_{nl}-arc-pancyclic regular tournaments
   - Extension V: arc-pandashcyclic regular tournaments
   - Extension VI: quasi_{ps}-arc-pancyclic regular tournaments

Digraph, path, cycle, k-strong

- $D = (V(D),\ A(D))$: a finite digraph D with the vertex-set $V(D)$ and the arc-set $A(D)$ without multiple arcs and loops.
- $xy \in A(D)$: $x \to y$ or x dominates y or y is dominated by x.

paths in digraphs	:	directed paths
cycles in digraphs	:	directed cycles
a t-cycle	:	a cycle of length t

- A digraph D is strong if for any two vertices x, y of D, there is a path from x to y and a path from y to x.

Semicomplete digraphs, tournaments

- A digraph D is semicomplete if for any two different vertices x and y of D there is at least one arc between them.
- A digraph D is called tournament if for any two different vertices x and y of D there is exactly one arc between them,
  or
  a tournament is an orientation of a complete graph,
  or
  a tournament is a semicomplete digraph without cycles of length 2.

Hamiltonian path/cycle in tournaments

Theorem 1 (Rédei, 1934)

Every tournament contains a Hamiltonian path.

- Hamiltonian path of a digraph D: a path containing all vertices of D.

Theorem 2 (Camion, 1959)

Every strong tournament contains a Hamiltonian cycle.

- Hamiltonian cycle of a digraph D: a cycle containing all vertices of D.

Two most famous theorems on tournaments

Theorem 3 (Moon, 1966)

Every strong tournament is vertex-pancyclic.

Theorem 4 (Alspach, 1967)

Every regular tournament is arc-pancyclic.

- A vertex or an arc of a digraph D on $n \geq 3$ vertices is called pancyclic if it is contained in a k-cycle for all $k \in \{3, \ldots, n\}$.
- A digraph D is called vertex-pancyclic (arc-pancyclic, respectively) if every vertex (every arc, respectively) of D is pancyclic.
- A digraph D is regular if there is an integer k such that $d^+(x) = d^-(x) = k$ for every vertex $x \in V(D)$.

An example of semicomplete multipartite digraphs

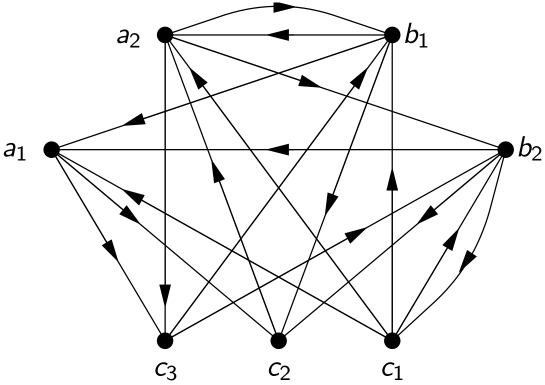

Figure : A semicomplete 3-partite digraph

Semicomplete multipartite digraphs (SMD's), multipartite tournaments (MT's)

- A semicomplete c-partite digraph D consists of c disjoint vertex sets $V_1, V_2, ..., V_c$ such that for every pair x, y of vertices, the following conditions are satisfied:
  (1) x and y are non-adjacent, if $x, y \in V_i$, $1 \le i \le c$;
  (2) there is at least one arc between x and y, if $x \in V_i$ and $y \in V_j$ with $i \ne j$, $1 \le i, j \le c$.
- A semicomplete multipartite digraph without a cycle of length 2 is called a multipartite tournament.
- Tournaments $\subset$ Multipartite tournaments $\subset$ Semicomplete multipartite digraphs,

since a tournament with n vertices is an n-partite tournament.

k-cycle in semicomplete multipartite digraphs

Theorem 5 (Bondy, 1976)

(1) *Every strong semicomplete c-partite ($c \geq 3$) digraph contains a k-cycle for all* $k \in \{3, 4, ..., c\}$.

(2) *If D is a strong semicomplete c-partite ($c \geq 5$) digraph, in which each partite set has at least two vertices, then D contains a k-cycle for some* $k > c$.

Problem 6 (Bondy, 1976)

Let D be a strong c-partite ($c \geq 5$) tournament, in which each partite set has at least 2 vertices.
Does D contain a $(c+1)$*-cycle ?*

A counterexample was found by Gutin in 1982, Balakrishnan and Paulraja in 1984.

$(c+1)$-cycles in c-partite tournaments

Theorem 7 (Guo and Volkmann, 1996)

Let D be a strong c-partite ($c \geq 5$) tournament, each of whose partite sets has at least 2 vertices. Then D has no $(c+1)$-cycle if and only if D is isomorphic to a member of W_m, where $m-1$ is the diameter of D.

Problem 8 (Guo and Volkmann, 1996)

Give a characterization of strong semicomplete c-partite ($c \geq 5$) digraphs, in which each partite set has at least 2 vertices and there is no $(c+2)$-cycle or $(c+k)$-cycle for some $k \geq 3$.

Theorem 9 (Yeo, 1997)

Every regular c-partite tournament with $c \geq 5$ is vertex-pancyclic.

Extension of pancyclicities to SMD's I: quasi $_{\mathrm{p}}$-pancyclicities

Theorem 10 (Goddard and Oellermann, 1991)

Every vertex of a strong semicomplete c-partite ($c \geq 3$) digraph is in a cycle that contains vertices from exactly k partite sets for all k with $3 \leq k \leq c$,

- A quasi $_{\mathrm{p}}$-k-cycle (quasi $_{\mathrm{p}}$-$(k-1)$-path, respectively) in a semicomplete multipartite digraph is a cycle (path, respectively) which contains vertices from exactly k different partite sets.
- A vertex (an arc, respectively) in a semicomplete c-partite ($c \geq 3$) digraph is quasi $_{\mathrm{p}}$-pancyclic, if it lies on a quasi $_{\mathrm{p}}$-k-cycle for all $3 \leq k \leq c$.

Generalization of Moon's theorem to SMD's I: quasi $_{\mathrm{p}}$-vertex-pancyclicity

Theorem 10 (Goddard and Oellermann, 1991) *Every strong semicomplete c-partite ($c \geq 3$) digraph is quasi $_{\mathrm{p}}$-vertex-pancyclic.*

Regular multipartite tournaments

Lemma 11

Let D be a c-partite tournament with partite sets $V_1, V_2, ..., V_c$. If D is regular, then $|V_1| = |V_2| = \cdots = |V_c|$.

Proof: Let x_i be a vertex in V_i for $i = 1, 2, ..., c$. Since D is regular, there exists an integer k such that

$$k = d^+(x_i) = d^-(x_i) = \frac{1}{2}\sum_{j \neq i} |V_j| \quad \text{for } i = 1, 2, ..., c.$$

It follows that $|V_1| = |V_2| = \cdots = |V_c|$. □

Generalization of Alspach's theorem to MT's I: quasi$_{\text{p}}$-arc-pancyclicity

Theorem 12 (Guo and Kwak, 1998)

Let D be a regular c-partite tournament with $c \geq 3$. If the cardinality common to all partite sets of D is odd, then D is quasi$_{\text{p}}$-arc-pancyclic.

Extension of pancyclicities to SMD's II: quasi $_{l}$-pancyclicities

Theorem 13 (Gutin, 1993)

If D is a strong semicomplete c-partite ($c \geq 3$) digraph, and if v is the only vertex in one of the partite sets, then D has a t-cycle containing v for every t with $3 \leq t \leq c$.

Theorem 14 (Guo and Volkmann, 1994)

Let D be a strong c-partite ($c \geq 3$) tournament. Then every partite set of D contains at least one vertex which lies on a t-cycle for all $t \in \{3, 4, ..., c\}$.

- A vertex (an arc, respectively) in a semicomplete c-partite ($c \geq 3$) digraph is quasi $_{l}$-pancyclic, if it lies on a t-cycle for all $3 \leq t \leq c$.

Generalization of Moon's theorem to SMD's II: quasi$_{|}$-vertex-pancyclicity

Theorem 14 (Guo and Volkmann, 1994) *Every partite set of a strong c-partite ($c \geq 3$) tournament contains at least one quasi$_{|}$-pancyclic vertex.*

Generalization of Alspach's theorem to MT's II: quasi$_{\mathrm{l}}$-arc-pancyclicity

Theorem 15 (Guo & Kwak, 1998)

Let T be a regular c-partite tournament. If every arc of T is in a 3-cycle, then each arc of T is also in a k-cycle for all $3 \le k \le c$, i.e., T is quasi$_{\mathrm{l}}$-arc-pancyclic.

Extension of pancyclicities to SMD's III: quasi $_o$-pancyclicities

Definition 16 (Guo, 1996)

An outpath of a vertex x (an arc xy, respectively) in a digraph D is a path starting at x (xy, respectively) such that x dominates the endvertex of the path only if the endvertex also dominates x in D.

Note that in a tournament, a vertex x (an arc xy, respectively) is in a cycle of length k if and only if x (xy, respectively) has an outpath of length $k-1$.

Extension of pancyclicities to SMD's III: quasi $_o$-pancyclicities

- A vertex (an arc xy, respectively) in a semicomplete c-partite ($c \geq 3$) digraph is quasi $_o$-pancyclic, if it lies on an outpath of length $k-1$ for all $k \in \{3, 4, ..., c\}$.
- A semicomplete c-partite ($c \geq 3$) digraph D is quasi $_o$-vertex-pancyclic (quasi $_o$-arc-pancyclic, respectively), if every vertex (arc, respectively) of D is quasi $_o$-pancyclic.

Generalization of Moon's theorem to SMD's III: quasi $_\circ$-vertex-pancyclicity

Theorem 17 (Guo, 1996)

Every strong semicomplete c-partite ($c \geq 3$) digraph is quasi$_\circ$-vertex-pancyclic.

Generalization of Alspach's theorem to MT's III: quasi $_o$-arc-pancyclicity

Theorem 18 (Guo, 1996)

Every regular c-partite ($c \geq 3$) tournament is quasi $_o$-arc-pancyclic.

Theorem 19 (Zhang and Zhou, 1997)

Let T be an almost regular c-partite ($c \geq 8$) tournament. If each partite set of T has at least 2 vertices, then every arc of T has an outpath of length $k-1$ for all $k \in \{4, ..., c\}$.

Theorem 20 (Yeo, 1998)

Let T be an almost regular c-partite ($c \geq 8$) tournament with $|V(D)| \geq 107$. Then every arc of T has an outpath of length $k-1$ for all $k \in \{4, 5, ..., |V(D)|\}$.

Extension of pancyclicities to SMD's IV: quasi $_{nl}$-pancyclicities

Question (Guo, 1994): Let D be a strong semicomplete c-partite ($c \geq 3$) digraph with $|V(D)| = n \geq 3$.

$$\min_{v \in V(D)} |\{\ell \geq 3 | v \text{ lies on an } \ell\text{-cycle in } D\}| \geq f(c,n) =?$$

Theorem 21 (Guo, Pinkernell and Volkmann, 1997)

Let D be a strong semicomplete c-partite ($c \geq 3$) digraph. Then every vertex of D is in a k-cycle or $(k+1)$-cycle for all $k \in \{3, 4, \ldots, c\}$.

Answer (Guo, 2021): $c - 2$.

Extension of pancyclicities to SMD's IV: quasi $_{nl}$-pancyclicities

Theorem 22 (Guo and Surmacs, 2022)

Every vertex of a strong semicomplete c-partite digraph D, where $c \geq 3$, belongs to $c-2$ cycles whose lengths are at least 3 and are pairwise distinct.

- A vertex (an arc xy, respectively) in a semicomplete c-partite ($c \geq 3$) digraph is quasi $_{nl}$-pancyclic, if it lies on at least $c-2$ cycles, whose lengths are at least 3 and pairwise distinct.
- A semicomplete c-partite ($c \geq 3$) digraph D is quasi $_{nl}$-vertex-pancyclic (quasi $_{nl}$-arc-pancyclic, respectively), if every vertex (arc, respectively) of D is quasi $_{nl}$-pancyclic.

Generalization of Moon's theorem to SMD's IV: quasi $_{\mathsf{nl}}$-vertex-pancyclicity

Theorem 22 (Guo and Surmacs, 2022) *Every strong semicomplete c-partite ($c \geq 3$) digraph is quasi$_{\mathsf{nl}}$-vertex-pancyclic.*

Generalization of Alspach's theorem to MT's IV: quasi $_{\mathrm{nl}}$-arc-pancyclicity

Conjecture 1 (Guo, 2022)

Every regular c-partite tournament with $c \geq 3$ is quasi $_{\mathrm{nl}}$-arc-pancyclic.

Extension of pancyclicities to SMD's V: pandashcyclicities

Definition 23 (Guo, 2022)

Let D be a semicomplete c-partite ($c \geq 3$) digraph with n vertices. A dashcycle $C := v_1v_2\cdots v_tv_1$ of length t, where $3 \leq t \leq n$ and $v_i \in V(D)$ for $1 \leq i \leq t$, is a t-cycle or consists of some vertex disjoint paths of D, which satisfies the following conditions:

(1) v_{i+1} dominates v_i only if v_i also dominates v_{i+1} in D for $i = 1, 2, \ldots, t$, where $v_{t+1} = v_1$;

(2) For $3 \leq t \leq c+1$, C has at least $t-1$ arcs, and for $c+2 \leq t \leq n$, C has at least c arcs.

Note that in a tournament, a dashcycle of length t is a t-cycle.

Extension of pancyclicities to SMD's V: pandashcyclicities

Definition 24 (Guo, 2022)

Let D be a semicomplete c-partite $(c \geq 3)$ digraph with n vertices. A dashpath $P := v_1 v_2 \cdots v_t$ of length $t-1$, where $3 \leq t \leq n$ and $v_i \in V(D)$ for $1 \leq i \leq t$, is a path of length $t-1$ or consists of some vertex disjoint paths of D, which satisfies the following conditions:

(1) v_{i+1} dominates v_i only if v_i also dominates v_{i+1} in D for $i = 1, 2, \ldots, t-1$;

(2) For $3 \leq t \leq c+1$, C has at least $t-2$ arcs, and for $c+2 \leq t \leq n$, C has at least $c-1$ arcs.

Note that in a tournament, a dashpath of length $t-1$ is a $(t-1)$-path.

Extension of pancyclicities to SMD's V: pandashcyclicities

- A vertex (an arc xy, respectively) in a semicomplete c-partite ($c \geq 3$) digraph D is pandashcyclic, if it lies on a dashcycle of length t for all $t \in \{3, 4, \ldots, |V(D)|\}$.
- A semicomplete c-partite ($c \geq 3$) digraph D is vertex-pandashcyclic (arc-pandashcyclic, respectively), if every vertex (arc, respectively) of D is pandashcyclic.

Generalization of Moon's theorem to SMD's V: vertex-pandashcyclicity

Theorem 25 (Guo, 2022)

Every strong semicomplete c-partite ($c \geq 3$) digraph is vertex-pandashcyclic.

Generalization of Alspach's theorem to MT's V: arc-pandashcyclicity

Theorem 26 (Guo, 2022)

Every regular c-partite ($c \geq 3$) tournament is arc-pandashcyclic.

Extension of pancyclicities to SMD's VI: quasi $_{\rm ps}$-arc-pancyclicity

Definition 27

Let $D = (V_1 \uplus V_2 \uplus \cdots \uplus V_c, A)$ be a semicomplete c-partite ($c \geq 3$) digraph.

- An arc $x_i x_j$ of D with $x_i \in V_i$ and $x_j \in V_j$ is called quasi $_{\rm ps}$-pancyclic in D, if for all $t \in \{3, \ldots, c\}$, D has a path from V_j to V_i, which contains vertices from exactly t partite sets.

Background for introducing quasi $_{\mathrm{ps}}$-pancyclic arc

Corollary 28 (Bang-Jensen, Maddaloni and Simonsen, 2013)

There is a polynomial algorithm for deciding whether a multipartite tournament D with given distinct partite sets V_i, V_j contains a quasi $_{\mathrm{p}}$-Hamiltonian path between some vertex $x \in V_i$ and some vertex $y \in V_j$.

- To decide whether a multipartite tournament D contains a quasi $_{\mathrm{p}}$-Hamiltonian path between two prescribed vertices:

$\mathcal{NP}$-complete.

Generalization of Alspach's theorem to MT's VI: quasi $_{\rm ps}$-arc-pancyclicity

Conjecture 2 (Guo, 2022)

Every regular c-partite tournament with $c \geq 3$ is quasi $_{\rm ps}$-arc-pancyclic.

Remark: quasi $_{\rm p}$-arc-pancyclicity $\Longrightarrow$ quasi $_{\rm ps}$-arc-pancyclicity

Thanks for your attention!